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What is Many-Valued Logic ?

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Abstract

Firstly we examine the definition of many-valued logic within the framework of (logical) matrix theory. Secondly we discuss the general result, challenging the existence of many-valued logic, according to which every logic may be seen as two-valued. Thirdly we analyze the principle of bivalence and show that it appears at a deeper level than one usually thinks.

0. Many questions[1]

Many-valued logic is a very important field of research in logic. But what is many-valued logic ? One starting point of many-valued logic is that not "everything is true or false" (principle of bivalence), that there are other truth-values. But what does this mean exactly ?

Shall many-valued logic be thought from the point of view of matrix theory, following the pioneer works of Lukasiewicz and Post ? And from this point of view, is there a coherent definition of what is *a* many-valued logic ? Is a many-valued logic any logic that cannot be characterized by a two-valued matrix but can be characterized by some other matrix ?

Is the principle of bivalence not preserved at a deeper level in a (for example) three-valued matrix, with the distinction between designated and non-designated values? And can we challenge the principle of bivalence even at this level ?

Suszko provided a bivalent semantics for Lukasiewicz's three-valued logic. This result is in fact part of a general reduction theorem according to which any logic has a bivalent semantics. What is the exact meaning of this theorem ?

1. Many-valued logic and matrix theory

We can think that many-valued logic is the study of many-valued logics. In this case, in order to answer the question "What is many-valued logic ?", we have to answer the question "What is a many-valued logic ?".

To answer this second question we must be able to have an answer to the question "What is a logic ?". We will consider here a logic as a language together with a set of tautologies, that is to say, mathematically speaking, a *logic* L is a structure $<\mathcal{F};\mathbb{T}>$ where $\mathcal{F}$ is an absolute free algebra (algebra of formulas) and $\mathbb{T}$ a subset of the domain $\mathbb{F}$ of the algebra[2].

Such a structure is a particular case of what is called a (logical) *matrix*, i.e. an algebra together with a subset of its domain, elements of this subset being called *designated values*. A matrix is called a *k-valued* matrix according to the cardinality k of its domain (i.e. the domain of its algebra).

Given a logic L, a matrix M is said to *characterized* L if the underlying algebras are of the same type and if, for every homomorphism from the algebra of formulas to the algebra of the matrix, the value of every tautology is a designated value of M.

A logic is said to be a *k-valued logic* when it can be characterized by a k-valued matrix but by no i-valued matrix ($i<k$) and :

A *many-valued logic* is a k-valued logic ($2<k$). [DF1]

Maybe one can think that now the job is over, that everything is alright and that we have a good definition for eternity.

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[1] The origin of the reflexions developed here is a review-essay (cf. [da Costa/Béziau/Bueno 1996]) of Malinowki's books on many-valued logic (cf. [Malinowski 1993]). However the present paper is self-contained and does not require the readings of the book and its review. Acknowledgments are due to the co-authors of the review (N.C.A. da Costa and O.A.S.Bueno), to G.Malinowski with whom these questions were discussed., as well as to two anonymous referees We would like also to thank M.V.Kritz for his invitation to work in the LNCC.

[2] The definition of the set of formulas as an absolute free algebra is the mathematical characterization of a zero-order (i. e. sentential) language due to the Polish school. We will not consider here first-order logic, but what we will say can be adapted to it as well as to the case where a consequence relation is taken instead of a set of tautologies.

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However the situation is not that simple, for example, intuitonistic logic, modal logics, and so on, are many-valued logics according to this definition, and even if one is generous enough to accept in the realm of many-valued logic such nice logics he will perhaps be afraid when he will see that the "so on" above includes in fact almost every logic.

This is a direct consequence of a generalization of one of the famous Lindenbaum's theorems, according to which *every structural logic can be charcaterized by a matrix* (in fact a k-valued matrix, where k is the cardinality of the language of the logic)[3]. But what is a *structural logic* ? It is a logic where the set of tautologies is stable under substitutions, i.e. stable under endomorphisms of the algebra of formulas.[4] Most people think that a logic must be structural.[5] Thus according to the first definition of a many-valued logic and to Lindenbaum's theorem they should think that every logic which is not two-valued is many-valued.[6] They are very generous indeed.

To try to solve this paradoxical conclusion, let us examine the case of intuitionistic logic. Gödel (cf. [Gödel 1932]) has shown that it cannot be characterized by a finite matrix. Maybe that is one reason to say that it is not many-valued, thinking of this second definition :

A *many-valued logic* is a k-valued logic ($2<k<\aleph_0$). [DF2]

But in this case Lukasiewicz's infinite many-valued logic should also be excluded from the realm of many-valued logic. However Lukasiewicz's infinite many-valued logic is generated by a matrix which is not the canonical one (i.e. the one given by Lindenbaum's theorem).[7] Thus perhaps the reason why intuitionistic logic is not considered as a many-valued logic is that :

(1) apart from the canonical matrix given by Lindenbaum's theorem, there is no intuitive (infinite) matrix which characterized it,

(2) it is better characterized by other semantics (Kripke's semantics for example).

It would be difficult, accordingly to (1), to write down a precise definition due to the fuzziness of the concept of "intuitive matrix", and moreover maybe someone can find one day an intuitive infinite matrix for intuitionistic logic ; in fact Jaskowski (cf. [Jaskowski 1936]) has proved a close result : he has shown that intuitionistic logic can be characterized by a "intuitive" denumerable class of finite matrices.

As far as (2) is concerned, it is clear that the study of intuitionistic logic does not reduce to the use of (logical) matrix theory ; but is this a good reason to consider that it is not a many-valued logic ?

In fact there is a kind of tendency which leads one to consider that many-valued logic is basically a part of matrix theory. This can be easily explained by the fact that matrix theory is one of the main tools for the study of many-valued logic. However this is not necessarily the only tool for the study of a many-valued logic. The difficulty is that when studying a many-valued logic (for example in the sense of DF2) with other tools, its specific character of being many-valued may not appear anymore as a key feature. On the other hand, k-valued ($2<k$) matrices can be usefully used even in the case of bivalent logic, as shown by a result such as the one of Bernays (cf. [Bernays 1926]) about independency of axioms for classical logic.

Thus maybe it will be better to say that a logic can be viewed, treated or studied from the viewpoint of many-valuedness, rather than to say that it is a many-valued logic. In this case we can try to answer the question "What is many-valued logic ?" trying to answer "What is many-valuedness ?" rather than "What is a many-valued logic ?". Many-valued logic taken as many-valuedness appears then as a bunch of technics which can be applied in the study of logic : it can be classical set theory[8], classical proof theory[9], fuzzy logic[10] or a logic like the three-valued logic of Lukasiewicz.

But can we say that these technics reduce to (part of) matrix theory ?

2. Every logic can be seen as two-valued

R.Suszko (cf. [Suszko 1975]) has constructed a two-valued semantics for Lukasiewicz's three-valued logic. How has he succeeded to realize this impossible task ?[11]

[3] This result can be extended in some way to the case of a consequence relation, but in this case a unique matrix is not enough, a class of matrices is necessary, known under the name "Lindenbaum bundle" (details about these results can be found in [Wojcicki 1988]).

[4] These defintions of structurality and substitution are due to Los and Suszko (cf. [Los and Suszko 1958]).

[5] This concept of structurality is in fact the mathematical characterization of the invariability of the form with regards to the contents (the meaning), according to which logic is said to be "formal".

[6] The limit cases of a 1-valued matrix, a 0-valued matrix, a matrix without designated values, etc., are commonly excluded.This is what we have implicitly done.

[7] It is in fact generated by two intuitive matrices since it has been shown that the denumerable and the continous Lukasiewicz 's matrices determine one and the same logic.

[8] The method of boolean-valued models can be considered part of many-valuedness.

[9] Many-valuedness is used in connection with the problem of cut-elimination (cf. e.g. [Girard 1976]).

[10] The exact relation between fuzzy logic and many-valued logic is still to be carefully examined (some people think for example that they are the same). Anyway it seems that, at least, technics from many-valued logic are used in fuzzy logic.

[11] This result is still not well-known and often not taken in account in standard presentations of many-valued logic. [Malinowski 1993] presents and discusses this result that nobody should ignore.

To understand this paradox we have to make a distinction between different kinds of semantics. Should necessarily a two-valued semantics for a logic L be a two-valued characterized matrix for this logic ? Imagine that we have a set B of functions from F to {0,1} and that this set is not the set of homomorphisms between the algebra of formulas and a corresponding matrix's algebra. Imagine however that this set is an adequate semantics for L in the sense that a formula of L is a tautology iff its value under every function of B is 1. What kind of semantics is this ? Why not call it a two-valued semantics? This is such kind of semantics that Suszko did provide for Lukasiewicz's three-valued logic.

Suszko (cf. [Suszko 1977]) makes the distinction between *logical valuation* and *algebraic valuation*. What he calls algebraic valuations are functions whose source is the algebra of formulas and which are homomorphims between this algebra and an algebra of similar type (an algebra of a matrix). And what he calls logical (bi)valuations are functions from the algebra of formulas into the set {0,1} which are not homomorphisms (between the algebra of formulas and an algebra of similar type).

Following Suszko's distinction, we will call a *logical semantics* a semantics made of logical valuations and a *matricial semantics* a semantics made of algebraic valuations[12].

It is possible to provide for every logic a two-valued logical semantics.[13] The difficulty here is the same as in the case of the canonical Lindenbaum matrix : this general (trivial) result does not allow us to say that the study of a given logic is necessarily interesting from a bivalent viewpoint ; we must find a "good', "intuitive", "fruitful" two-valued logical semantics. It is not clear for example that Suszko's two-valued semantics for Lukasiewicz's three-valued logic is really relevant for its study. It has not yet been used to prove significant new results about it.

Two-valued logical semantics is not a tool which has been widely used in Poland, where the matrix approach prevails[14]. But it has been successfully and extensively used in Brazil, at first for developing an "intuitive" semantics for paraconsistent logic (see [da Costa/Alves 1977]) and further for the development of other logics and even as a basic tool for the systematization of logic. With regards to this last aim, it has been called by N.C.A. da Costa, the *theory of valuation* (for details on this theory see [da Costa/Béziau 1994a & 1994b]).

J.-Y.Béziau (cf. [Béziau 1990]) has presented a three-valued semantics (for the paraconsistent logic C1) which is not a matricial semantics but a logical semantics in the same sense as the above definition of two-valued logical semantics, i.e. this semantics is a set of functions whose target is a set of three elements (one being designated) which are not algebraic valuations.

The interest of doing so is that the use of this three-valued logical semantics lets appear some interesting features which do not appear with the use of the standard two-valued logical semantics for this logic.

We can reasonably think that k-valued logical semantics (2<k) are part of many-valuedness and therefore that the answer to the question of the end of the first section is negative[15] : many-valuedness does not reduce to (part of) matrix theory.

3. Many-valued logic and the principle of bivalence

There is a much more informal and general definition of many-valued logic which is the following :

A *many-valued* logic is a logic which transgresses the principle of bivalence. [DF3]

This definition is related to the main motivation for many-valued logic according to which not everything is true or false, that they are other truth-values. It is worth noting that this definition does not refer to a special kind of semantics, like the preceding ones.[16]

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[12] We will not use the expression "algebraic semantics" because generally it is used to denote something else, for example the quotient algebra of a logic. (Is an algebraic semantics in this last acceptation part of many-valuedness ? This is an interesting question, but we will not discuss it here)

[13] There are several ways more or less trivial to do so. In the case of a consequence relation, it must obey, at least, the three standard axioms of Tarski. This result has been stated independently by R.Suszko and N.C.A. da Costa in the 1970's, however none of them wrote it down explicitly, so that there is no historical reference to it. Moreover it seems that there are some differences between the two logicians, at least if we follow G.Malinowski's presentation of Suszko's result according to which structurality seems to be required, which is not the case in da Costa's result. For more details and comments about this reduction's theorem see [Béziau 1995] and [da Costa/Béziau/Bueno 1996].

[14] This clearly appears looking at [Wojcicki 1988] which is supposed to be a kind of systematic exposition of the work of the Polish school : matrix theory is discussed at lenght but two-valued logical semantics is not presented explictly.

[15] Suszko in [Suszko 1977] does not consider k-valued logical semantics (2<k) for philosophical reasons. For him the values of a matrix must not be considered as logical values, in particular because he thinks that the multiplication of logical values is a "mad idea", that there are only two logical values, truth and falsity. We are not arguing here that the additional logical values in a k-valued logical semantics (2<k) have to be taken as logical values (our terminology is just a matter of convention) whatever this could mean, but we are stressing the fact that this kind of semantics can be technically useful in the study of logic and that the vocable "many-valued" naturally applies to them.

[16] In fact the formulation of the principle of bivalence goes back to Antiquity, at a time where, for example, the above distinction between logical and matricial semantics was not thinkable. The fact that this principle is fundamental in logic is credited to Chrysippus, that is why

The principle of bivalence can be expressed in the following way:

Every sentence is either true or false but not both.

It must be pointed out that this definition is quite general and in particular does not involve negation. This is not always clear because some people see the principle of bivalence as the conjunction of the principle of excluded middle and the principle of contradiction and these principles are often considered as principle ruling negation.

For example Moisil (in [Moisil 1977], p.34) presents the following definition of the principle of bivalence :

"No sentence is at the same time true and false, this is the principle of contradiction ; every sentence is true or false and there is no third possibility, this is the principle of excluded middle. We will say that a sentence can have one of the two logical values : truth or falsity. This statement constitutes the principle of bivalence".

It is important to stress that the above definitions of the principles of excluded middle and contradiction do not either involve negation and in particular are not equivalent to the facts that respectively $p \vee \neg p$ and $\neg(p \wedge \neg p)$ must be tautologies.

It is clear that it is possible to construct a logic in which there are sentences which are neither true nor false (which is therefore a many-valued logic in the sense of [DF3]) but in which $p \vee \neg p$ is a tautology. And it is also possible to construct a logic in which some sentences are both true and false but in which $\neg(p \wedge \neg p)$ is a tautology.

Also it is interesting to note that a logic in which some sentences are both true and false can sometimes be viewed as a logic in which no sentences are both true and false but in which a sentence can be neither true nor false: the attribution of both truth and falsity to a sentence can be seen as the attribution of a unique third value[17].

The expression "a logic in which there are some sentences which are neither true nor false" is in fact ambiguous at least for two reasons.

The first point lies in the relations between logic and semantics. If we consider a logic as a set of tautologies (or a consequence relation), due to the reduction's theorem, it can always be considered from the point of view of a bivalent semantics in which every sentence is either true or false. However this same logic may also be considered from the point of view of a semantics (matricial or not) in which some sentences are neither true or false.

To avoid confusion we must therefore distinguish between a logic taken alone and a logic taken with a specific semantics.

Following our previous conclusions, it seems reasonable to consider part of many-valuedness the study of a logic from the point of view of a semantics according to which there are sentences which are neither true nor false even if a matricial or logical two-valued semantics can be associated to it.

The second difficulty is about what we must understand by true and false. When we have for example a (logical or matricial) semantics with three values 0, ½, 1 and that each of these three values can be attributed at least to one sentence, does it mean necessarily that there is a sentence which is neither true nor false ?

Here we must be very carefull not to be abused by plays of words. Shall we consider, as it is commonly done, that true and false are two of these values (respectively 1 and 0) and that ½ is neither truth nor falsity ? Or shall we consider that truth encompasses the set of all designated values and falsity the set of all non-designated values ?[18]

This duality between designated and non-designated values is clearly a version of the principle of bivalence as G.Malinowski pointed out (cf. e.g. [Malinowski 1993]).

The definition of the set of tautologies (or of a consequence relation) is based on this duality. Inversely the notion of set of tautologies (or of consequence relation) is fundamentally bivalent as shown by the fact that it can always be defined by a two-valued (logical) semantics.

Logic is thus bivalent rooted deeper than one usually imagines. Going beyond this genuine bivalency means a lot more than constructing three-valued matrices, it means throwing away our conception of logic as a set of tautologies (or as a consequence relation). A first step in this direction was taken by G.Malinowski with the introduction of the concept of *inferential many-valuedness* (see [Malinowski 1994 & 1996]).[19]

Moisil used to call "Logiques non-chrysippiennes" what is nowadays standardly called "many-valued logic" or "multiple-valued logic".

[17] This is the case of Priest's paraconsistent logic LP (introduced in [Priest 1979]). In one semantical presentation there are only two truth-values and some sentences can be true and false at the same time, in another semantical presentation there is a third value which sticks to the former truth-false sentences.

[18] There are some people, not making the distinction between these two possibilities, using the words "truth' and "falsity" simultaneously with two different meanings. This leads to confusion: for example G.Priest takes on the one hand truth to be only 1 to say that his negation is a contradictory forming operator and on the other hand truth to be 1 and ½ to define his logic in order that it appears as a paraconsistent logic (For a detailed discussion of this matter see [Béziau/Schireson 1996]).

[19] Malinowski constructs (using an extended concept of matrix) a consequence relation which has no two-valued logical semantics because it fails to obey the "identity" axiom of Tarski. However it has been shown (cf. [Krause/Béziau 1997]) that we can adapt in some way two-valued logical semantics even in the case of such kind of consequence relation.

4. Conclusion

Let us briefly summarize the conclusions of our analysis :

1. There is no satisfactory answer (even within matrix theory) to the question "What is a many-valued logic ?"

2. Many-valued logic is a bunch of technics which can be used in logic and that can be denoted under the name "many-valuedness".

3. Many-valuedness does not reduce to (part of) matrix theory. Logical many-valued semantics can be considered as part of many-valuedness.

4. Matricial and logical many-valued semantics preserve at a deeper level two-valuedness, in the definition of truth, with the division between designated and non-designated values. It is possible to develop many-valuedness in a more radical way.

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